arXiv:math/0602346v1 [math.ST] 16 Feb 2006

Goodness-of-Fit Tests for Symmetric Stable Distributions – Empirical Characteristic Function Approach

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October, 2005

Abstract

We consider goodness-of-fit tests of symmetric stable distributions based on weighted integrals of the squared distance between the empirical characteristic function of the standardized data and the characteristic function of the standard symmetric stable distribution with the characteristic exponent α estimated from the data. We treat α as an unknown parameter, but for theoretical simplicity we also consider the case that α is fixed. For estimation of parameters and the standardization of data we use maximum likelihood estimator (MLE) and an equivariant integrated squared error estimator (EISE) which minimizes the weighted integral. We derive the asymptotic covariance function of the characteristic function process with parameters estimated by MLE and EISE. For the case of MLE, the eigenvalues of the covariance function are numerically evaluated and asymptotic distribution of the test statistic is obtained using complex integration. Simulation studies show that the asymptotic distribution of the test statistics is very accurate. We also present a formula of the asymptotic covariance function of the characteristic function process with parameters estimated by an efficient estimator for general distributions.

1 Introduction.

The family of stable distributions is one of the most important classes of distributions in probability theory. The general central limit theorem asserts that if a suitably normalized sum of independently and identically distributed (i.i.d.) random variables has a limit distribution, only possible limits are the stable distributions (Chapter 6 of Feller (1971)). Concerning statistical inference, because of their attractive properties such as heavy tails, many models based on stable distributions have been considered in both social and natural sciences (Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999), Rachev and Mittnik (2000)). Therefore it is important to consider goodness-of-fit tests of stable distributions. However few researches on goodness-of-fit tests of stable distributions have been conducted due to the difficulty in expressing their density functions explicitly. The purpose of this paper is to propose goodness-of-fit tests based on the empirical characteristic function, since the characteristic functions of stable distributions are explicitly given. For past researches on goodness-of-fit tests of heavy-tailed distributions using empirical characteristic function approach, see Gürtler and Henze (2000) and Matsui and Takemura (2005). Both papers treat Cauchy ($\alpha = 1$) distribution which is one of the stable distributions.

Let $f(x; \mu, \sigma, \alpha)$ denote the symmetric stable density with the characteristic function

$$\Phi(t) = \exp(i\mu t - |\sigma t|^{\alpha}),$$

where the parameter space is

$$\Omega = \{-\infty < \mu < \infty,\ \sigma > 0,\ 0 < \alpha \le 2\}.$$

Here α is the characteristic exponent, μ is the location parameter and σ is the scale parameter. For the standard case $(\mu, \sigma) = (0, 1)$ we simply write the characteristic function as $\Phi(t; \alpha) = \exp(-|t|^\alpha)$ and the density function as $f(x; \alpha)$. In this parameterization stable distributions form a location-scale family for each value of α, i.e.,

$$f(x; \mu, \sigma, \alpha) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma}; \alpha).$$

In order to cope with more general situation or for notational convenience we also write the parameters as

$$\theta = (\theta_1, \theta_2, \theta_3) = (\mu, \sigma, \alpha)$$

and write corresponding density, distribution or characteristic function as

$$f(x; \mu, \sigma, \alpha) = f(x; \theta), \quad F(x; \mu, \sigma, \alpha) = F(x; \theta), \quad \Phi(x; \mu, \sigma, \alpha) = \Phi(x; \theta).$$

Here we note that θ is a vector.

In this paper we often differentiate functions of the parameter θ and the data x with respect to x, μ, σ and α. Since we will consider affine invariant (location-scale invariant) tests, it is often sufficient to evaluate the derivatives at the standard case $(\mu, \sigma) = (0, 1)$. For example we use the notation

$$f_\mu(x; \alpha) = \frac{\partial}{\partial \mu} f(x; \mu, \sigma, \alpha)_{|(\mu,\sigma)=(0,1)} \quad \text{or} \quad f'(x; \alpha) = \frac{\partial}{\partial x} f(x; \mu, \sigma, \alpha)_{|(\mu,\sigma)=(0,1)}.$$

Concerning the characteristic function we also use $\nabla_\theta \Phi(t; \theta) = (\Phi_\mu(t; \theta), \Phi_\sigma(t; \theta), \Phi_\alpha(t; \theta))$ where, for example,

$$\Phi_\mu(t; \theta) = \frac{\partial}{\partial \mu} \Phi(t; \mu, \sigma, \alpha).$$

For standard case $(\mu, \sigma) = (0, 1)$ we write

$$\Phi_\mu(t; \alpha) = \frac{\partial}{\partial \mu} \Phi(t; \mu, \sigma, \alpha)_{|(\mu,\sigma)=(0,1)}.$$

Given a random sample $x_1, \ldots, x_n$ from an unknown distribution F, we want to test the null hypothesis H_1 that F belongs to the family of stable distributions $f(x; \mu, \sigma, \alpha)$ and the null hypothesis H_2 that F belongs to the family of stable distributions $f(x; \mu, \sigma, \alpha)$ with $\alpha = \alpha_0$ fixed. Note that $H_1 \supset H_2$. Here we explain our proposed procedure for testing H_1, because for H_2 we can simply replace $\hat{\alpha}$ by α_0.

As remarked above stable distributions form a location scale family and we consider affine invariant tests. The proposed tests are based on the difference between the empirical characteristic function

$$\Phi_n(t) = \Phi_n(t; \hat{\mu}, \hat{\sigma}) = \frac{1}{n} \sum_{j=1}^{n} \exp(ity_j), \qquad y_j = \frac{x_j - \hat{\mu}}{\hat{\sigma}}, \tag{1.1}$$

of the standardized data y_j and the characteristic function with α estimated from the data

$$\Phi(t) = \Phi(t; \hat{\alpha}) = e^{-|t|^{\hat{\alpha}}}.$$

Here $\hat{\mu} = \hat{\mu}_n = \hat{\mu}_n(x_1, \ldots, x_n)$, $\hat{\sigma} = \hat{\sigma}_n = \hat{\sigma}_n(x_1, \ldots, x_n)$ and $\hat{\alpha} = \hat{\alpha}_n = \hat{\alpha}_n(x_1, \ldots, x_n)$ are affine equivariant estimators of μ, σ, α satisfying

$$\begin{aligned}
\hat{\mu}_n(a + bx_1, \ldots, a + bx_n) &= a + b\hat{\mu}_n(x_1, \ldots, x_n), \\
\hat{\sigma}_n(a + bx_1, \ldots, a + bx_n) &= b\hat{\sigma}_n(x_1, \ldots, x_n), \\
\hat{\alpha}_n(a + bx_1, \ldots, a + bx_n) &= \hat{\alpha}_n(x_1, \ldots, x_n),
\end{aligned}$$

for all $-\infty < a < \infty$ and $b > 0$.

As equivariant estimators we consider maximum likelihood estimator (MLE) and an equivariant integrated squared error estimator (EISE) defined in (2.6) below. The reason for considering MLE is its asymptotic efficiency and the reason for EISE is that its definition is similar to our proposed test statistic.

Following Gürtler and Henze (2000) and Matsui and Takemura (2005) we propose the following test statistic

$$D_{n,\kappa} := n \int_{-\infty}^{\infty} \left|\Phi_n(t) - e^{-|t|^{\hat{\alpha}}}\right|^2 w(t)dt, \qquad w(t) = e^{-\kappa|t|},\ \kappa > 0. \tag{1.2}$$

$D_{n,\kappa}$ is the weighted L^2-distance between $\Phi_n(t)$ and the characteristic function $e^{-|t|^{\hat{\alpha}}}$ of $f(x; \hat{\alpha})$ with respect to the weight function $w(t) = e^{-\kappa|t|}$, $\kappa > 0$. This weight function is chosen for convenience, so that we can evaluate the asymptotic covariance function of the empirical characteristic function process under H_1.

The test statistic $D_{n,\kappa}$ has an alternative representation, which is useful for obtaining its asymptotic distribution.

$$D_{n,\kappa} = \int_{-\infty}^{\infty} \left|\hat{Z}_n(t)\right|^2 \hat{\sigma}_n w(\hat{\sigma}_n t)dt, \tag{1.3}$$

where

$$\hat{Z}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left\{\cos(tx_j) + i\sin(tx_j) - e^{-|\hat{\sigma}_n t|^{\hat{\alpha}_n}} \left(\cos(t\hat{\mu}_n) + i\sin(t\hat{\mu}_n)\right)\right\}. \tag{1.4}$$

$\hat{Z}_n(t)$ corresponds to the empirical characteristic function process.

Our test statistic $D_{n,\kappa}$ is a quadratic form of the empirical characteristic function process. Although we derive an explicit form of the asymptotic covariance function of the empirical characteristic function process, it is not trivial to derive the asymptotic distribution of $D_{n,\kappa}$ under H_1 and H_2 from the covariance function, especially when the parameters are estimated. See chapter 7 of Durbin (1973a) and Durbin (1973b) for tests based on empirical distribution functions with estimated parameters and see Gürtler and Henze (2000) and Matsui and Takemura (2005) for tests based on empirical characteristic functions. As in Matsui and Takemura (2005) we evaluate the asymptotic distribution of $D_{n,\kappa}$ for the case MLE by numerically approximating the eigenvalues of the asymptotic covariance function. By numerically evaluating the asymptotic distribution we can also check the convergence of the finite sample distributions which we obtain by Monte Carlo simulations.

Concerning EISE, as shown below, the asymptotic covariance function of the empirical characteristic function process is very complicated. Furthermore we found that Monte Carlo simulation involving EISE is very time consuming. Therefore in this paper we show theoretical results on our proposed test statistic involving EISE and leave numerical studies to our subsequent works.

This paper is organized as follows. In Section 2.1 we first define and summarize properties of MLE and EISE. Then in Section 2.2 we state theoretical results on asymptotic distribution of $D_{n,\kappa}$ under H_1 in Theorem 2.3 and Theorem 2.5 and results under H_2 as the corollaries of these theorems. Numerical evaluations of asymptotic critical values of $D_{n,\kappa}$ under H_1 and H_2 for MLE are discussed in Section 3. Simulation studies of MLE and corresponding test statistic $D_{n,\kappa}$ are given in Section 4, including the study of finite sample power behavior in Section 4.3.

2 Main results

2.1 Estimators and their asymptotic properties

For our purposes we need asymptotic covariance matrices and "asymptotically linear representations" (AL representations) of the estimators. We describe asymptotic properties of maximum likelihood estimator (MLE) following DuMouchel (1973). We also define an equivariant integrated squared error estimator (EISE) and give asymptotic properties of EISE. For MLE explicit expressions of the asymptotic covariance matrix and AL representations are given in the Cauchy case $\alpha = 1$.

As shown in DuMouchel (1973), MLE is asymptotically normal and asymptotically efficient. The likelihood equation is given by

$$\frac{\partial L}{\partial \mu} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \frac{1}{2\pi f(\frac{x_j-\mu}{\sigma};\alpha)} \int_{-\infty}^{\infty} e^{-it\left(\frac{x_j-\mu}{\sigma}\right)} \Phi_\mu(t;\alpha)dt = 0, \tag{2.1}$$

$$\frac{\partial L}{\partial \sigma} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \frac{1}{2\pi f(\frac{x_j-\mu}{\sigma};\alpha)} \int_{-\infty}^{\infty} e^{-it\left(\frac{x_j-\mu}{\sigma}\right)} \Phi_\sigma(t;\alpha)dt = 0, \tag{2.2}$$

$$\frac{\partial L}{\partial \alpha} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} \frac{1}{2\pi f(\frac{x_j-\mu}{\sigma};\alpha)} \int_{-\infty}^{\infty} e^{-it\left(\frac{x_j-\mu}{\sigma}\right)} \Phi_\alpha(t;\alpha)dt = 0, \tag{2.3}$$

where

$$(\Phi_\mu(t;\alpha), \Phi_\sigma(t;\alpha), \Phi_\alpha(t;\alpha)) = \left(ite^{-|t|^\alpha}, -e^{-|t|^\alpha}|t|^\alpha\alpha, -e^{-|t|^\alpha}|t|^\alpha \log|t|\right).$$

EISE is an affine equivariant version of the ISE (integrated squared error) estimator proposed by Paulson *et al.* (1975). The original ISE estimator of Paulson *et al.* (1975) is not equivariant. Robustness and efficiency of ISE estimators of location and scale parameters are discussed in Thornton and Paulson (1977) for the normal case and in Besbeas and Morgan (2001) for the Cauchy case. EISE is based on the standardized empirical characteristic function. Let

$$\Phi_n(t;\mu,\sigma) = \frac{1}{n}\sum_{j=1}^{n} \exp\left(it\frac{x_j-\mu}{\sigma}\right),$$

which is the same as (1.1) with $\hat{\mu}_n$ and $\hat{\sigma}_n$ replaced by μ and σ. Write

$$Q(\mu,\sigma,\alpha) = \int_{-\infty}^{\infty} \left|\Phi_n(t;\mu,\sigma) - e^{-|t|^\alpha}\right|^2 w(t)dt, \tag{2.4}$$

where we use the following weight function

$$w(t) = \exp(-\nu|t|^{\bar{\alpha}}), \quad \nu > 0. \tag{2.5}$$

Here we call $\bar{\alpha}$ weighting index and ν weighting constant. EISE $(\hat{\mu}_n, \hat{\sigma}_n, \hat{\alpha}_n)$ is defined to be the minimizer of $Q(\mu, \sigma, \alpha)$:

$$Q(\hat{\mu}_n, \hat{\sigma}_n, \hat{\alpha}_n) = \min_{\mu,\sigma,\alpha} Q(\mu, \sigma, \alpha). \tag{2.6}$$

It is easy to see that EISE is affine equivariant by definition. Note that the weighting constant κ in the test statistic (1.2) and the weighting constant ν in (2.5) for EISE may be different. In our theoretical results on EISE we can treat more general weighting functions, i.e., $w(t) \geq 0$ is an arbitrary even function. However for performing goodness-of-fit tests, it seems natural to set $\alpha_0 = \bar{\alpha}$ and $\nu = \kappa$. The integral $Q(\theta)$ can be calculated as

$$Q(\theta) = \int_{-\infty}^{\infty} \left\{ \frac{1}{n^2} \sum_{j,k}^{n} \cos\big(t(x_j - x_k)/\sigma\big) - \frac{2}{n} \sum_{j=1}^{n} \cos\big(t(x_j - \mu)/\sigma\big) e^{-|t|^\alpha} + e^{-2|t|^\alpha} \right\} w(t) dt.$$

The estimators satisfy the following estimating equations $0 = \partial Q/\partial\mu = \partial Q/\partial\sigma = \partial Q/\partial\alpha \Leftrightarrow 0 = Q_\mu(\theta) = Q_\sigma(\theta) = Q_\alpha(\theta)$.

$$Q_\mu(\theta) = -\int_{-\infty}^{\infty} \left\{ \frac{1}{n} \sum_{j=1}^{n} \sin\big(t(x_j - \mu)/\sigma\big) t e^{-|t|^\alpha} \right\} w(t) dt, \tag{2.7}$$

$$\begin{aligned} Q_\sigma(\theta) = \frac{1}{2} \int_{-\infty}^{\infty} \Bigg[& \left\{ \frac{1}{n^2} \sum_{j,k=1}^{n} \cos\big(t(x_j - x_k)\big) - \frac{2}{n} \sum_{j=1}^{n} \cos\big(t(x_j - \mu)\big) e^{-|\sigma t|^\alpha} + e^{-2|\sigma t|^\alpha} \right\} \\ & \times \{ w'(\sigma|t|)\sigma|t| + w(\sigma t) \} + 2 \left\{ \frac{1}{n} \sum_{j=1}^{n} \cos\big(t(x_j - \mu)\big) - e^{-|\sigma t|^\alpha} \right\} \alpha |\sigma t|^\alpha e^{-|\sigma t|^\alpha} w(\sigma t) \Bigg] dt, \end{aligned} \tag{2.8}$$

$$Q_\alpha(\theta) = \int_{-\infty}^{\infty} \left\{ \frac{1}{n} \sum_{j=1}^{n} \cos\big(t(x_j - \mu)/\sigma\big) - e^{-|t|^\alpha} \right\} e^{-|t|^\alpha} |t|^\alpha \log|t| w(t) dt, \tag{2.9}$$

where $w'(x) = dw(x)/dx$. Note that in case of $Q_\sigma(\theta)$ differentiation was done after the transformation $t \to \sigma t$.

In the rest of this paper we use the following notations. $\xrightarrow{D}$ means weak convergence of random variables or stochastic processes, $\xrightarrow{P}$ means convergence in probability.

The asymptotically linear representations (AL representation) give an method of approximating asymptotic behavior of the estimator by sum of functions of i.i.d. random samples. For the standard symmetric stable case $f(x; \alpha)$ we need following three expressions,

$$\begin{aligned} \sqrt{n}\hat{\mu}_n &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} l_1(X_j) + r_{1n}, \\ \sqrt{n}(\hat{\sigma}_n - 1) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} l_2(X_j) + r_{2n}, \\ \sqrt{n}(\hat{\alpha}_n - \alpha) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} l_3(X_j) + r_{3n}, \qquad r_{1n}, r_{2n}, r_{3n} \xrightarrow{P} 0. \end{aligned}$$

For the case of MLE, AL representations are given in terms of the score functions ((2.1-2.3)) and the Fisher information matrix. The proof is standard and omitted.

Theorem 2.1 *Let $I(\theta)$ denote the Fisher information matrix*

$$I(\theta) = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & I_{23} \\ 0 & I_{32} & I_{33} \end{pmatrix}, \qquad I_{ij}(\theta) = -E_\theta \left[\frac{\partial^2 \log f(x,\theta)}{\partial\theta_i \partial\theta_j} \right].$$

The AL representations $l_\theta(x) = (l_\mu(x), l_\sigma(x), l_\alpha(x))$ at the standard case $(\mu,\sigma) = (0,1)$ are given by $l_\theta(x) = I^{-1}(\theta) h_\theta(x)$, where $h_\theta(x)$ are

$$h_\mu(x) = \frac{1}{2\pi f(x;\alpha)} \int_{-\infty}^{\infty} ite^{-itx} e^{-|t|^\alpha} dt, \tag{2.10}$$

$$h_\sigma(x) = -\frac{\alpha}{2\pi f(x;\alpha)} \int_{-\infty}^{\infty} e^{-itx} e^{-|t|^\alpha} |t|^\alpha dt, \tag{2.11}$$

$$h_\alpha(x) = -\frac{1}{2\pi f(x;\alpha)} \int_{-\infty}^{\infty} e^{-itx} e^{-|t|^\alpha} |t|^\alpha \log|t| dt. \tag{2.12}$$

Concerning EISE we can employ standard theory of U-statistics. The proof is given in Appendix A.

Theorem 2.2 *Define a 3×3 symmetric matrix*

$$A = A(\alpha) = A(\theta)_{|(\mu,\sigma)=(0,1)} \tag{2.13}$$

by

$$\begin{aligned}
A_{12} &= A_{12} = 0, \\
A_{11} &= \int_{-\infty}^{\infty} e^{-2|t|^\alpha} t^2 w(t) dt, \\
A_{22} &= \alpha^2 \int_{-\infty}^{\infty} e^{-2|t|^\alpha} |t|^{2\alpha} w(t) dt, \\
A_{23} &= \alpha \int_{-\infty}^{\infty} e^{-2|t|^\alpha} |t|^{2\alpha} \log|t| w(t) dt, \\
A_{33} &= \int_{-\infty}^{\infty} e^{-2|t|^\alpha} |t|^{2\alpha} (\log|t|)^2 w(t) dt,
\end{aligned}$$

and define $h_\theta(x) = (h_\mu(x), h_\sigma(x), h_\alpha(x))'$ by

$$h_\mu(x) = \int_{-\infty}^{\infty} t \sin(tx) e^{-|t|^\alpha} w(t) dt, \tag{2.14}$$

$$h_\sigma(x) = -\alpha \int_{-\infty}^{\infty} (\cos(tx) - e^{-|t|^\alpha}) e^{-|t|^\alpha} |t|^\alpha w(t) dt, \tag{2.15}$$

$$h_\alpha(x) = -\int_{-\infty}^{\infty} (\cos(tx) - e^{-|t|^\alpha}) e^{-|t|^\alpha} |t|^\alpha \log|t| w(t) dt. \tag{2.16}$$

For EISE the AL representations $l_\theta(x) = (l_\mu(x), l_\sigma(x), l_\alpha(x))$ at the standard case $(\mu,\sigma) = (0,1)$ are given by

$$l_\theta(x) = A(\alpha)^{-1} h_\theta(x) \tag{2.17}$$

and their asymptotic covariance matrix at the standard case is given by

$$A^{-1}E[h_\theta(X)h_\theta'(X)]A^{-1\prime} = A^{-1}\begin{pmatrix} H_{\mu\mu} & 0 & 0 \\ 0 & H_{\sigma\sigma} & H_{\sigma\alpha} \\ 0 & H_{\alpha\sigma} & H_{\alpha\alpha} \end{pmatrix} A^{-1\prime} = \begin{pmatrix} J_{11} & 0 & 0 \\ 0 & J_{22} & J_{23} \\ 0 & J_{32} & J_{33} \end{pmatrix},$$

where each element of $H = E[h_\theta(X)h_\theta'(X)]$ *is*

$$\begin{aligned}
H_{\mu\mu} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(e^{-|s-t|^\alpha} - e^{-|s+t|^\alpha})\right\} e^{-(|s|^\alpha+|t|^\alpha)} st \ w(s)w(t)dsdt, \\
H_{\sigma\sigma} &= \alpha^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(e^{-|s-t|^\alpha} + e^{-|s+t|^\alpha}) - e^{-(|s|^\alpha+|t|^\alpha)}\right\} e^{-(|s|^\alpha+|t|^\alpha)}|st|^\alpha w(s)w(t)dsdt, \\
H_{\sigma\alpha} &= \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(e^{-|s-t|^\alpha} + e^{-|s+t|^\alpha}) - e^{-(|s|^\alpha+|t|^\alpha)}\right\} e^{-(|s|^\alpha+|t|^\alpha)}|st|^\alpha \log|t| w(s)w(t)dsds, \\
H_{\alpha\alpha} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{2}(e^{-|s-t|^\alpha} + e^{-|s+t|^\alpha}) - e^{-(|s|^\alpha+|t|^\alpha)}\right\} e^{-(|s|^\alpha+|t|^\alpha)}|st|^\alpha \log|s|\log|t| w(s)w(t)dsdt.
\end{aligned}$$

Note that the above AL representations and asymptotic matrices involve definite integrals, which require numerical integration. But for some special cases like Cauchy ($\alpha = 1$) we can calculate several integrals analytically. Analytic expressions are useful for checking correctness of numerical calculations concerning Theorem 2.1. We give the following Corollary for the case of $\alpha = 1$ and MLE.

Corollary 2.1 *Let* $\gamma \doteq 0.577216$ *denote Euler constant. In the Cauchy case* ($\alpha = 1$) *and MLE, at* $(\mu, \sigma) = (0, 1)$*, the AL representations are given as* $l_\theta(x) = I^{-1}h_\theta(x)|_{\alpha=1}$*, where*

$$\begin{aligned}
h_\mu(x) &= \frac{2x}{x^2+1}, \qquad h_\sigma(x) = \frac{x^2-1}{x^2+1}, \\
h_\alpha(x) &= \frac{1-x^2}{x^2+1}\left[\frac{1}{2}\log(x^2+1) - 1 + \gamma\right] + \frac{2x}{x^2+1}\arctan x,
\end{aligned}$$

$$I_{11} = I_{22} = \frac{1}{2}, \quad I_{23} = I_{32} = \frac{1}{2}(1-\gamma-\log 2), \quad I_{33} = \frac{1}{2}\left\{\frac{\pi^2}{6} + (\gamma + \log 2 - 1)^2\right\}.$$

A similar result is given in Section 6 of Matsui and Takemura (2006) and the proof is omitted.

2.2 Asymptotic theory of the proposed test statistics

In this section theoretical results on asymptotics of the proposed test statistic $D_{n,\kappa}$ are obtained. From another expression of $D_{n,\kappa}$ (1.3) we derive weak convergence of $\hat{Z}_n(t)$ and weak convergence of test statistic $D_{n,\kappa}$ in the following two theorems. These results correspond to those of Cauchy case stated in Matsui and Takemura (2005) where parameter $\alpha = 1$ is fixed. As a special case we also describe the Cauchy case involving estimation of α in the corollary below. Furthermore a general formula of asymptotic covariance function of the empirical characteristic process with parameters estimated by an efficient estimator is given in the latter part of this section. As already remarked several times, we can assume without loss of generality that $X_1, \ldots, X_n$ is random sample from $f(x;\alpha)$ because of affine invariance of our tests. Following Gürtler and Henze (2000) we use the Fréchet space $C(\mathbf{R})$ of continuous functions on the real line $\mathbf{R}$ for considering the random processes. The metric of $C(\mathbf{R})$ is given by

$$\rho(x, y) = \sum_{j=1}^{\infty} 2^{-j}\frac{\rho_j(x,y)}{1+\rho_j(x,y)},$$

where $\rho_j(x,y) = \max_{|t|\le j} |x(t) - y(t)|$.

We first give the asymptotic covariance function of the empirical characteristic function process with parameters estimated by MLE and EISE. In the following theorems the elements of the inverse of the Fisher information matrix $I(\theta)$ and the matrix A in (2.13) are denoted with superscripts I^{ij} and A^{ij}.

Theorem 2.3 *Let $X_1, \ldots, X_n$ be* i.i.d. *$f(x;\alpha)$ random variables and let $\hat{Z}_n$ be defined in (1.4). Then $\hat{Z}_n \xrightarrow{D} Z$ in $C(\mathbf{R})$, where Z is a zero mean Gaussian process with covariance functions given below.*

MLE :

$$\Gamma(s,t) = e^{-|t-s|^\alpha} - e^{-(|t|^\alpha+|s|^\alpha)} - \Big\{ I^{11} st + I^{22} |st|^\alpha \alpha^2 + I^{23} |st|^\alpha \alpha \log|st| + I^{33} |st|^\alpha \log|s| \log|t| \Big\} e^{-(|t|^\alpha+|s|^\alpha)}. \tag{2.18}$$

EISE :

$$\begin{aligned}\Gamma(s,t) &= e^{-|t-s|^\alpha} - e^{-(|t|^\alpha+|s|^\alpha)} \\ &+ \{J_{11} st + J_{22}\alpha^2 |st|^\alpha + J_{23}\alpha |st|^\alpha (\log|s| + \log|t|) + J_{33}|st|^\alpha \log|t| \log|s|\} e^{-(|t|^\alpha+|s|^\alpha)} \\ &+ \big\{ (B_\sigma A^{22} + B_\alpha A^{23}) \alpha (|t|^\alpha + |s|^\alpha) + (B_\sigma A^{23} + B_\alpha A^{33})(|t|^\alpha \log|t| + |s|^\alpha \log|s|) \big\}\, e^{-(|t|^\alpha+|s|^\alpha)} \\ &- A^{11} t e^{-|t|^\alpha} \int_{-\infty}^{\infty} e^{|s-u|^\alpha - |u|^\alpha} u\, w(u) du[2] \\ &- \alpha (A^{22}\alpha + A^{23} \log|t|) |t|^\alpha e^{-|t|^\alpha} \int_{-\infty}^{\infty} e^{|s-u|^\alpha - |u|^\alpha} |u|^\alpha w(u) du[2] \\ &- (A^{23}\alpha + A^{33} \log|t|) |t|^\alpha e^{-|t|^\alpha} \int_{-\infty}^{\infty} e^{|s-u|^\alpha - |u|^\alpha} |u|^\alpha \log|u| w(u) du[2]\end{aligned} \tag{2.19}$$

where [2] *after a term means symmetrization with respect to s and t, i.e., $g(s,t)[2] = g(s,t) + g(t,s)$, and*

$$B_\alpha = \int_{-\infty}^{\infty} e^{-2|u|^\alpha} |u|^\alpha \log|u| w(u) du, \qquad B_\sigma = \alpha \int_{-\infty}^{\infty} e^{-2|u|^\alpha} |u|^\alpha w(u) du.$$

As a corollary the asymptotic covariance function for the case of fixed α is given as follows.

Corollary 2.2 *Under the same conditions of Theorem 2.3, when α is fixed, $\hat{Z}_n(t) \xrightarrow{D} Z$ in $C(\mathbf{R})$, where Z is a zero mean Gaussian process with covariance functions given below.*

MLE :

$$\Gamma(s,t) = e^{-|t-s|^\alpha} - e^{-(|t|^\alpha+|s|^\alpha)} - (I^{11} st + I^{22} |st|^\alpha \alpha^2) e^{-(|t|^\alpha+|s|^\alpha)}.$$

EISE :

$$\begin{aligned}\Gamma(s,t) &= e^{-|t-s|^\alpha} - e^{-(|t|^\alpha+|s|^\alpha)} + \{J_{11} st + J_{22}\alpha^2 |st|^\alpha + A^{22} B_\sigma \alpha (|t|^\alpha + |s|^\alpha)\} e^{-(|t|^\alpha+|s|^\alpha)} \\ &- A^{11} t e^{-|t|^\alpha} \int_{-\infty}^{\infty} e^{-|s-u|^\alpha - |u|^\alpha} u w(u) du[2] - A^{22}\alpha^2 |t|^\alpha e^{-|t|^\alpha} \int_{-\infty}^{\infty} e^{-|s-u|^\alpha - |u|^\alpha} |u|^\alpha w(u) du[2].\end{aligned}$$

The following corollary gives the asymptotic covariance function when the true distribution is Cauchy $C(\mu, \sigma)$, but the characteristic exponent α is estimated by MLE.

Corollary 2.3 *Let $X_1, \ldots, X_n$ be* i.i.d. *$C(0,1)$ random variables and let $\hat{Z}_n$ be defined in (1.4), where parameters are estimated by MLE. Then $\hat{Z}_n \xrightarrow{D} Z$ in $C(\mathbf{R})$, where Z is a zero mean Gaussian process with covariance function given below.*

$$
\begin{aligned}
(2.20) \qquad \Gamma_c(s,t) &= e^{-|t-s|} - \{1 + 2(st + |st|)\}e^{-|s|-|t|} \\
&\quad - \frac{12}{\pi^2}\{\log|s| + (\gamma + \log 2 - 1)\}\{\log|t| + (\gamma + \log 2 - 1)\}|st|e^{-|s|-|t|}.
\end{aligned}
$$

Furthermore for efficient estimations including MLE, we can derive more general result after some formulations. We assume parameter space Θ is p-dimensional. First, we define an efficient estimator $\hat{\theta}_n$ of θ_0 as such that

$$
\begin{aligned}
\sqrt{n}(\hat{\theta}_n - \theta_0) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} I^{-1}(\theta_0)\frac{\partial f(X_j;\theta_0)}{\partial\theta} + \epsilon_n \\
(2.21) \qquad &:= \frac{1}{\sqrt{n}}\sum_{j=1}^{n} l_E(X_j;\theta_0) + \epsilon_n,
\end{aligned}
$$

where $\epsilon_n \xrightarrow{P} 0$. This definition coincides with formula (20) of Durbin (1973b), though his definition is given for nuisance parameters. He also gives conditions that the estimator $\hat{\theta}_n$ satisfies formula (2.21) under more general arguments including local alternatives (see the condition (A3) of Durbin (1973b)). Note that Durbin (1973b) considers estimation only for nuisance parameters $\hat{\theta}_n$ including local alternatives and the parameters of interest (the null hypothesis) are not estimated. However in our case the whole parameters are estimated without nuisance parameters because the null hypothesis of H_1 is the whole parameter space.

Second, we modify the conditions (iv) of Csörgő (1983) as
(iv) $l_E(x;\theta)$ in (2.21) is a p-dimensional Borel measurable function, $E[l_E(X_1;\theta_0)] = (0,\ldots,0)$, and $I^{-1}(\theta_0) = E[l_E(X_1;\theta_0)l'_E(X_1;\theta_0)]$ is finite and positive definite.*

Theorem 2.4 *Let $X_1, \ldots, X_n$ be* i.i.d. *$F(x;\theta_0)$ random variables and let $k(x,t) = \cos(tx) + i\sin(tx)$. Consider the kernel transformed empirical characteristic process*

$$
(2.22) \qquad \hat{Z}_n = \int k(x,t)d\left\{\sqrt{n}\big(F_n(x) - F(x;\hat{\theta}_n)\big)\right\},
$$

where $F_n(x)$ denotes the empirical distribution function. Then $\hat{Z}_n \xrightarrow{D} Z$ in $C(\mathbf{R})$ under the conditions (i), (ii)*, (v) and (vi) of Csörgő (1983) where $l(\cdot;\theta_0)$ is replaced by $l_E(\cdot;\theta_0)$ and (iv)*. Here Z is a zero mean Gaussian process with covariance function*

$$
(2.23) \qquad \Gamma(s,t) = \Phi(s-t;\theta_0) - \Phi(s;\theta_0)\overline{\Phi(t;\theta_0)} - \nabla_\theta\Phi(s;\theta_0)'I^{-1}(\theta)\overline{\nabla_\theta\Phi(t;\theta_0)},
$$

where $\Phi(t;\theta_0)$ is the characteristic function and $\nabla_\theta\Phi(t;\theta_0)$ is the derivative of $\Phi(t;\theta_0)$ with respect to parameter vector θ.

Theorem 2.4 is interpreted as the Fourier kernel transformed version of Theorem 2 of Durbin (1973b) with nuisance parameters corresponds to the estimated null hypothesis. In our subsequent works we will consider extension of Theorem 2.4 to the case of local alternatives.

Finally we state the following theorem concerning the weak convergence of $D_{n,\kappa}$.

Theorem 2.5 *Under the conditions of Theorem 2.1*

$$D_{n,\kappa} = \int_{-\infty}^{\infty} \hat{Z}_n(t)^2 \hat{\sigma}_n e^{-\hat{\sigma}_n \kappa |\hat{\sigma}_n t|^{\hat{\alpha}_n}} dt \xrightarrow{D} D_\kappa := \int_{-\infty}^{\infty} Z(t)^2 e^{-\kappa |t|^\alpha} dt.$$

The proofs of the above theorems are given in Appendix B. We omit the proof of Theorem 2.5 since after obtaining Theorem 2.3 the proof is the same as that of Theorem 2.2 of Gürtler and Henze (2000).

3 Approximation of the asymptotic critical values of the proposed test statistics

In this section we investigate the distribution of D_κ for MLE. We briefly explain how to obtain the characteristic function of D_κ. Detailed treatments of this approach in statistical applications are given in Tanaka (1996) or Anderson and Darling (1952). Since the characteristic function of D_κ contains infinite product of functions of eigenvalues which can not be evaluated analytically, we approximate eigenvalues by theory of homogeneous integral equations of the second kind and the associated Fredholm determinant. Then utilizing complex integration, we invert the characteristic function and obtain series representation of the distribution of D_κ. Detailed theoretical argument of inversion process is given in Slepian (1957). Actual computational approximations are given in the next section. At the end of this section we transform our kernels $\Gamma(s,t)$ on $\mathbf{R}^2$ to kernels $K(s,t)$ on $[-1,1]^2$ for convenience in numerical computation.

In this paper we omit consideration of D_κ of EISE since kernels $\Gamma(s,t)$ have definite integrals and we need many numerical approximations. On the other hand for the case of MLE we can utilize past researches on Fisher information in DuMouchel (1975), Nolan (2001) and Matsui and Takemura (2006) to confirm the accuracy of our computation. For $\kappa > 1$ we can use the following standard form of Mercer's theorem.

Theorem 3.1 (Mercer's Theorem, Chapter 5 of Hochstadt (1973)) *Let $K(s,t)$ be the kernel of a positive self-adjoint operator on $L^2[-1,1]$ and suppose that $K(s,t)$ is continuous in both variables. Then*

$$K(s,t) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} f_j(s) f_j(t), \qquad 0 < \lambda_1 \le \lambda_2 \le \cdots \uparrow \infty, \tag{3.1}$$

where λ_j is an eigenvalue and $f_j(t)$ is the corresponding orthonormal eigenfunction of the integral equation

$$\lambda \int_{-1}^{1} K(s,t) f(t) dt \quad = \quad f(s). \tag{3.2}$$

The series (3.1) *converges uniformly and absolutely to $K(s,t)$.*

If $\kappa \le 1$ we need to deal with kernels which are not continuous at two points $(-1,-1)$ and $(1,1)$. We can see discontinuity at $(-1,-1)$ and $(1,1)$ in Figures 1, 3 and 5 in the case $\kappa = 1$. On the other hand there is no discontinuity at these points for $\kappa = 2.5$ as shown in Figures 2, 4 and 6. However as in Anderson and Darling (1952) the following version of Mercer's theorem by Hammerstein (1927) is useful.

Theorem 3.2 *Suppose that the covariance function $K(s,t)$ of a Gaussian process is continuous except at $(-1,-1)$ and $(1,1)$ with $\partial K(s,t)/\partial s$ continuous for $|s|, |t| < 1, s \ne t$, and bounded in $|s| \le 1 - \epsilon$ for every $t \in [-1,1]$ and every $\epsilon > 0$. Then the right hand side of* (3.1) *converges uniformly in every domain in the interior of $[-1,1]^2$.*

We apply the above theorems to a continuous covariance function $K(s,t)$ of a zero mean continuous Gaussian process $Z(t)$, $-1 < t < 1$, with a finite trace $\int_{-1}^{1} K(t,t)dt < \infty$. Let $X_1, X_2, \ldots,$ be i.i.d. standard normal random variables. Then the series

$$Y(t) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} f_j(t) X_j$$

converges in the mean and with probability one for each $t \in (-1,1)$. Then $Y(t)$ is a Gaussian process with $\mathrm{E}Y(t) = 0$ and $\mathrm{E}[Y(t)Y(s)] = K(s,t)$. Thus $Y(t)$ defines the same stochastic process as $Z(t)$. Let

$$W^2 = \int_{-1}^{1} Y^2(t)dt = \int_{-1}^{1} \left\{ \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} f_j(t) X_j \right\}^2 dt = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} X_j^2. \tag{3.3}$$

The characteristic function of W^2 is given as

$$\mathrm{E}(e^{iuW^2}) = \mathrm{E}[\exp(iu \sum_{j=1}^{\infty} X_j^2/\lambda_j)] = \prod_{j=1}^{\infty} \mathrm{E}[\exp(iuX_j^2/\lambda_j)] = \prod_{j=1}^{\infty} (1 - 2iu/\lambda_j)^{-\frac{1}{2}}.$$

The characteristic function has an alternative expression $1/\sqrt{D(2it)}$ where $D(\lambda)$ is the associated Fredholm determinant

$$D(\lambda) = \prod_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right).$$

There are two problems in treating the characteristic function in the form of the Fredholm determinant. One is in the approximation of $D(\lambda)$ itself and the other is in the Levý's inversion formula.

In the case of stable distributions the Fredholm determinant can not be explicitly evaluated and approximation of $D(\lambda)$ is needed as in the Cauchy case in Matsui and Takemura (2005). We approximate $D(\lambda)$ by discretizing the homogeneous integral equation and approximating eigenvalues of resulting finite system of linear equations. Then the integral equation (3.2) is approximated by the following finite system of linear equations

$$\tilde{f} = \frac{\lambda}{N} \tilde{K} \tilde{f},$$

where

$$\tilde{K} = \begin{pmatrix} K(\xi_1, \xi_1) & \ldots & K(\xi_1, \xi_N) \\ \vdots & & \vdots \\ K(\xi_N, \xi_1) & \ldots & K(\xi_N, \xi_N) \end{pmatrix}, \qquad \tilde{f} = \begin{pmatrix} f(\xi_1) \\ \vdots \\ f(\xi_N) \end{pmatrix}.$$

Then the Fredholm determinant is approximated as

$$\tilde{D}_N(\lambda) = \left| I - \frac{\lambda}{N} \tilde{K} \right| = \prod_{j=1}^{N} \left(1 - \frac{\lambda}{\tilde{\lambda}_j}\right), \qquad 0 < \tilde{\lambda}_1 \le \cdots \le \tilde{\lambda}_N,$$

where $1/\tilde{\lambda}_j = 1/\tilde{\lambda}_j(N)$ are the eigenvalues of $\tilde{K}/N$. This method is called a quadrature method and we state a version of Theorem 3.4 of Baker (1977) concerning the convergence of eigenvalues.

Theorem 3.3 *Let the eigenvalues $\tilde{\lambda}_j(N)$ be obtained by the quadrature method. If $K(s,t)$ is positive definite and continuous in $s,t \in [-1,1]$,*

$$\lim_{N\to\infty} \tilde{\lambda}_j(N) = \lambda_j,$$

for each j and

$$\lim_{N\to\infty} \tilde{D}_N(\lambda) = D(\lambda)$$

for each λ.

Remark 3.1 *The covariance functions in* (3.7) *and* (3.8) *below do not satisfy the conditions of this theorem if $\kappa \leq 1$. However this theorem gives only a sufficient condition for the convergence. In our problem the values of $\tilde{D}_N(\lambda)$ seem to converge as we increase N even for the case $\kappa \leq 1$ and the resulting value is consistent with our Monte Carlo simulations. Therefore in the next section we use the approximation of this theorem even for the case $\kappa \leq 1$. It remains to theoretically prove that the approximation is valid for the case $\kappa \leq 1$.*

The probability density function of the proposed statistic is given by inverting the characteristic function $\Phi(t) = 1/\sqrt{D(2it)}$. Since integrand of inversion formula is often wildly oscillating and converges to 0 slowly, the ordinary numerical integration is difficult (Section 6.1 of Tanaka (1996)). However we can utilize theory of complex integration in Slepian (1957) and invert $\Phi(t)$ very efficiently. This method of inversion does not seem to be commonly implemented in statistical computations.

Assuming that the kernel $K(s,t)$ has no multiple eigenvalues and the number of the eigenvalues are infinite, the density and distribution of D_κ are calculated as

$$f_D(x) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k \int_{\frac{\lambda_{2k-1}}{2}}^{\frac{\lambda_{2k}}{2}} \frac{e^{-xy}}{\sqrt{\prod_{j=1}^{\infty}\left|1-\frac{2}{\lambda_j}y\right|}}dy,$$

$$F_D(x) = 1 - \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^k \int_{\frac{\lambda_{2k-1}}{2}}^{\frac{\lambda_{2k}}{2}} \frac{e^{-xy}}{y\sqrt{\prod_{j=1}^{\infty}\left|1-\frac{2}{\lambda_j}y\right|}}dy.$$

The series are alternating and convenient for checking convergence. Although the above representations have singularity at each endpoint of integral range, we can remove the singularity by the following transformation $y \mapsto z$ as in Slepian (1957). The k-th integral is transformed by

$$y = \frac{1}{2}\left(\frac{\lambda_{2k}}{2} - \frac{\lambda_{2k-1}}{2}\right)\cos \pi z + \frac{1}{2}\left(\frac{\lambda_{2k}}{2} + \frac{\lambda_{2k-1}}{2}\right), \quad 0 \leq z \leq 1. \tag{3.4}$$

Then

$$dy = -\pi\sqrt{\left(y - \frac{\lambda_{2k-1}}{2}\right)\left(\frac{\lambda_{2k}}{2} - y\right)}dz.$$

Hence we obtain the following representations suitable for numerical integration.

$$f_D(x) = \sum_{k=1}^{\infty}(-1)^k \int_0^1 \frac{e^{-xy}\sqrt{\left(\frac{\lambda_{2k}}{2} - y\right)\left(y - \frac{\lambda_{2k-1}}{2}\right)}}{\sqrt{\prod_{j=1}^{\infty}\left|1-\frac{2}{\lambda_j}y\right|}}dz, \tag{3.5}$$

$$F_D(x) = 1 - \sum_{k=1}^{\infty}(-1)^k \int_0^1 \frac{e^{-xy}\sqrt{\left(\frac{\lambda_{2k}}{2} - y\right)\left(y - \frac{\lambda_{2k-1}}{2}\right)}}{y\sqrt{\prod_{j=1}^{\infty}\left|1 - \frac{2}{\lambda_j}y\right|}} dz, \tag{3.6}$$

where y is given by formula (3.4).

Finally we will make a transformation of variable and map $\mathbf{R}^2$ into $[-1,1]^2$ in order to satisfy the finite interval condition of Mercer's theorem. This transformation also is useful for numerical approximation of eigenvalues. For deriving the distribution of D_κ, we have to incorporate the weight function $e^{-\kappa|t|}$ into the kernel, i.e., we consider the following kernel

$$\Gamma(s,t)e^{-\frac{\kappa}{2}(|s|+|t|)}.$$

Now we make the transformation $s \mapsto u$ defined by

$$s = -\operatorname{sgn} u \cdot \log(1 - |u|), \qquad -1 \le u \le 1.$$

Then

$$ds = \frac{1}{1-|u|}du.$$

The kernel and the eigenfunctions are transformed as

$$\begin{aligned} \Gamma(s,t) \quad &\mapsto \quad K(u,v) = \frac{\Gamma(-\operatorname{sgn} u \cdot \log(1-|u|), -\operatorname{sgn} v \cdot \log(1-|v|))}{\sqrt{(1-|u|)(1-|v|)}}, \\ f_j(s) \quad &\mapsto \quad \frac{f_j(-\operatorname{sgn} u \cdot \log(1-|u|))}{\sqrt{1-|u|}}. \end{aligned}$$

Eigenvalues of (3.2) do not change by this transformation and so does Fredholm determinant. After this transformation, writing s,t instead of u,v again, we have the following kernels on $[-1,1]^2$:

$$\begin{aligned} H_1 : K(s,t) \quad = \quad & \Big\{ e^{-|-\operatorname{sgn} s \cdot \log(1-|s|) + \operatorname{sgn} t \cdot \log(1-|t|)|^\alpha} - e^{-|\log(1-|s|)|^\alpha - |\log(1-|t|)|^\alpha} \\ & - \Big(I^{11} \operatorname{sgn} s \cdot \operatorname{sgn} t \cdot \log(1-|s|)\log(1-|t|) \\ & + \alpha^2 I^{22} |\log(1-|s|)\log(1-|t|)|^\alpha \\ & + \alpha I^{23} |\log(1-|s|)\log(1-|t|)|^\alpha \log|\log(1-|s|)\log(1-|t|)| \\ & + I^{33} |\log(1-|s|)\log(1-|t|)|^\alpha \log|\log(1-|s|)| \cdot \log|\log(1-|t|)| \Big) \\ & \quad \times e^{-|\log(1-|s|)|^\alpha - |\log(1-|t|)|^\alpha} \Big\} \{(1-|s|)(1-|t|)\}^{\frac{\kappa-1}{2}}. \end{aligned} \tag{3.7}$$

$$\begin{aligned} H_2 : K(s,t) \quad = \quad & \Big\{ e^{-|-\operatorname{sgn} s \cdot \log(1-|s|) + \operatorname{sgn} t \cdot \log(1-|t|)|^\alpha} - e^{-|\log(1-|s|)|^\alpha - |\log(1-|t|)|^\alpha} \\ & - \big(I^{11} \operatorname{sgn} s \cdot \operatorname{sgn} t \cdot \log(1-|s|)\log(1-|t|) + \alpha^2 I^{22} |\log(1-|s|)\log(1-|t|)|^\alpha \big) \\ & \quad \times e^{-|\log(1-|s|)|^\alpha - |\log(1-|t|)|^\alpha} \Big\} \{(1-|s|)(1-|t|)\}^{\frac{\kappa-1}{2}}. \end{aligned} \tag{3.8}$$

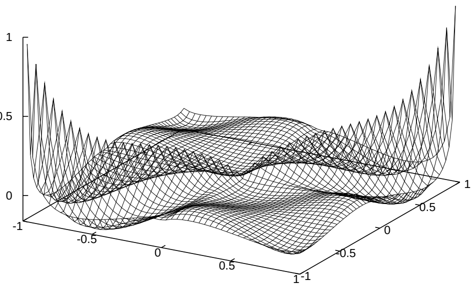

Figure 1: MLE-H1 ($\alpha = 1.0$, $\kappa = 1.0$)

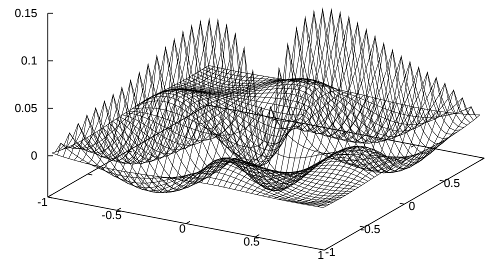

Figure 2: MLE-H1 ($\alpha = 1.0$, $\kappa = 2.5$)

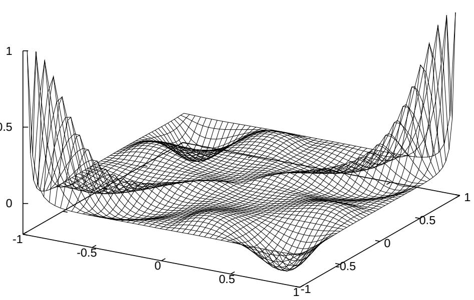

Figure 3: MLE-H1 ($\alpha = 1.5$, $\kappa = 1.0$)

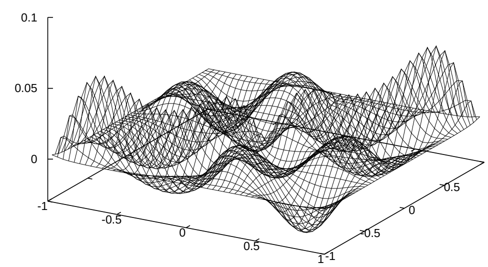

Figure 4: MLE-H1 ($\alpha = 1.5$, $\kappa = 2.5$)

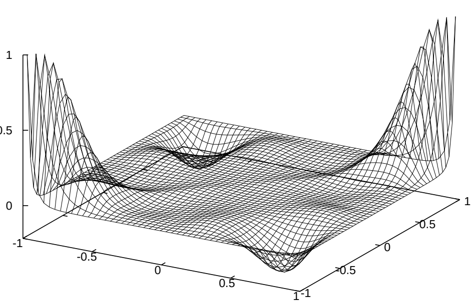

Figure 5: MLE-H1 ($\alpha = 1.8$, $\kappa = 1.0$)

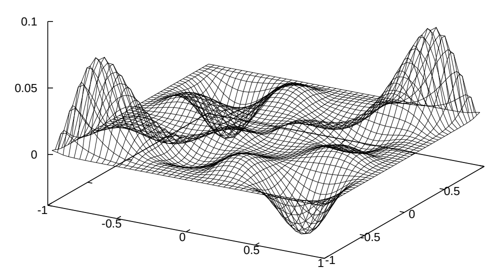

Figure 6: MLE-H1 ($\alpha = 1.8$, $\kappa = 2.5$)

3.1 Numerical approximation of critical values of D_κ

We approximate the eigenvalues in (3.6) by the quadrature method for the kernels (3.7) and (3.8). 800 eigenvalues are calculated by the above simple algorithm for the case of $\{\kappa = 1.0, 2.5, 5.0, 10.0\}$. We do not consider $\kappa = 0.1$ and 0.5 since convergence of infinite integral of $D_{n,\kappa}$ becomes very slow for small weights and we had numerical difficulties. In Matsui and Takemura (2005) the approximated sum of $\sum_{j=1}^{\infty} 1/\lambda_j$ did not converge to $\mathrm{E}[D_\kappa]$ first for small κ. We mention that, unlike the Cauchy $\alpha = 1$ case, we did not observe multiple eigenvalues for other symmetric stable distributions ($\alpha \neq 1$).

The infinite series and infinite products in (3.6) have to be approximated by a finite sum and finite products. Let l and m ($l < m$) denote the number of terms in the sum and the products respectively. Then we can approximate $F_D(x)$ as

$$F_D(x) \approx 1 - \sum_{k=1}^{l} (-1)^k \int_0^1 \frac{e^{-xy}\sqrt{\left(\frac{\lambda_{2k}}{2} - y\right)\left(y - \frac{\lambda_{2k-1}}{2}\right)}}{y\sqrt{\prod_{j=1}^{m}\left(1 - \frac{2}{\lambda_j} y\right)}} dz.$$

The series is alternating. Therefore the range of the critical value can be obtained by substituting lower bound of each positive term and upper bound of each negative term separately. However deriving analytical bound of integral of each term of series is difficult. Hence for accuracy of approximation of $F_D(x)$ we depend on numerical confirmation. First, we found that finite interval quadrature (QAG) is very accurate if we set relative error bounds below 10^{-5} and the convergence of series is very fast. The first 10 terms of series are enough to obtain 1% relative accuracy for most κ and x large enough to calculate critical values. Further for $m > 300$ the value of $F_D(x)$ does not change with m for most κ and large x. Finally the approximated percentage points (10% and 5 %) coincide with simulation results in Section 4.

We give Table 1 and Table 2 for approximate percentage points of D_κ under hypothesis H_1 and H_2 respectively. Intervals of $\alpha \in (0.5, 2.0)$ for H_1 and $\alpha \in (0.5, 2.0]$ are 0.1. In each table, we set $m = 500$ for $\kappa \leq 5.0$ and $m = 300$ for $\kappa = 10.0$, and set $l = 25$ for $\kappa \leq 2.5$ and $l = 10$ for $\kappa \geq 5.0$. Trial and error indicates that since D_κ for large κ is very small, we use only accurate large values of $1/\lambda_j$ among 800 values. We also plot the percentage points of each D_κ under H_1 in Figures 7-10. The values of percentages are continuous with respect to $\alpha \neq 2$. For large values of κ, percentage points of small α are large compared to that of large α.

4 Computational studies

In this section we give various computational results. Since the exact finite sample distributions are difficult to obtain, first we approximate the percentage points of D_κ under H_1 and H_2 respectively by Monte Carlo simulation. Then the power of testing H_2 for the finite sample is evaluated.

4.1 Maximum likelihood estimation

For MLE we maximize likelihood function in parameter space $-\infty < \mu < \infty$, $\sigma > 0$, $0 < \alpha \leq 2$ by utilizing the first derivatives of each parameter. Maximizations are done by the method based on M.J.D. Powell's TOLMIN from IMSL library. Although explicit forms of the density and the derivatives are not available for stable distributions, the method suggested by Matsui and Takemura (2006) which improves the original method of Nolan (1977,2001) gives very accurate approximations. We use median as the initial value for μ and for σ and α we do grid search and obtain the parameter values which maximize median

inserted log-likelihood among, say, 2000 points. Note that although we can set $\alpha = 1$ for the initial value of α, the convergence is slow compared with grid search based initial values. Base on 1000 Monte Carlo replications, the values of the estimators and the simulated information matrices $I(\hat{\theta}) = \mathrm{Cov}[\hat{\theta}_i, \hat{\theta}_j]^{-1}$ are given in Table 3 for $\alpha \in \{0.8, 1.0, 1.5, 1.8, 2.0\}$ and the sample size $n \in \{50, 100, 200\}$. We put true values at the upper row of each values of α in Table 3. Except for $\alpha = 2$, simulated values of $I(\hat{\theta})$ coincide with theoretical values $I(\theta)$ which are given in Matsui and Takemura (2006). Though information of α at $\alpha = 2.0$ is ∞ and asymptotic normality is not guaranteed (DuMouchel (1973)), we can estimate α computationally at $\alpha = 2$. Interestingly we observe $\hat{\alpha} = 2.0$ for 80%–90% of the cases for $n \geq 100$ and we also observe some downward bias.

4.2 Finite sample critical values of $D_{n,\kappa}$

We omit the case $\alpha = 2$ for both H_1 and H_2, since there are many papers concerning testing normality, e.g., Henze and Wagner (1997), Csörgő (1986,89) or Naito (1996). Further we investigate $D_{n,\kappa}$ only $\alpha = 1.0, 1.5, 1.8$ for convenience. More extensive simulation studies of $D_{n,\kappa}$ for other values of α are left to our future works.

We can compute $D_{n,\kappa}$ by (1.2), when the values of the estimators have converged. Based on 5000 Monte Carlo replications, the upper 10 and 5 percentage points of the statistics $D_{n,\kappa}$, $\kappa \in \{1.0, 2.5, 5.0, 10.0\}$ are tabulated in Tables 4, 5, 6, 7 for H_1 and Table 8, 9, 10, 11 for H_2. We tabulate simulated values in upper row and asymptotic values in lower row in box of each value of α. In the tables of H_1 when α and κ are large, the convergences of $D_{n,\kappa}$ to D_κ are slower than other values of α and κ. This tendency is also seen in the tables of H_2. This is explained as follows. Since behavior near origin of the characteristic function reflects behavior of the tail of distribution, the convergence of the empirical characteristic function near origin may be faster than that at distant values when the tail of distribution is heavy. However we find the values of $D_{n,\kappa}$ converging D_κ for all values of α and κ as $n \to \infty$.

4.3 Analysis of finite sample power

In this section we examine finite sample power under H_2. For alternative hypothesis we consider Student's t distribution with j degrees of freedom for $j = 1, 2, 3, 4, 5, 10, \infty$, ($t(1) = C(0,1)$ and $t(\infty) = N(0,1)$), since stable models are sometimes compared with Student's t models in empirical applications. We consider only $\alpha = 1.5$ and $\alpha = 1.8$ for null distribution of H_2 because the simulation studies are heavy when many values of α are considered. Investigations of other values of α are also left to our future works.

For the significance levels $\zeta = 0.1, 0.05$, finite sample power of the proposed tests are tabulated in Table 12 and 15, based on 1000 Monte Carlo replications. We summarize our findings. For $\alpha = 1.5$ the test with $\kappa = 10.0$ has poor power compared with other values of κ and the test with $\kappa = 5.0$ is the most powerful. When alternative hypothesis is $t(3)$ or near $t(3)$ finite sample power is not good. For $\alpha = 1.8$ the test with $\kappa = 10.0$ has also poor power as in the test for $\alpha = 1.5$. While the test with $\kappa = 5.0$ is the most powerful for heavy tail alternatives, the test with $\kappa = 2.5$ is more powerful for the light tail alternatives. These results are interesting because although the tails of $t(i)$ and $f(x;\alpha)$ are different, the distributions are not distinguishable well in finite samples.

Finally make a remark on how to perform a test of H_1. The problem is that even the asymptotic null distribution under H_1 depends on the true value of α. In the following remark $D^0_{n,\kappa}$ denotes the observed valued of the test statistic for H_1 and $D_{n,\kappa}(\xi;\alpha)$ denotes the upper ξ percentage points of the null distribution of $D_{n,\kappa}$ for H_1 when α is the true characteristic exponent.

Remark 4.1 *We can consider several procedures for H_1.*

1. *H_1 is rejected if $D^0_{n,\kappa} \geq \sup_{\alpha\in(0,2]} D_{n,\kappa}(\xi;\alpha)$,*

2. *We consider $\alpha \in [a,b]$ where $[a,b] \subset (0,2)$ is a fixed range and put $H_1 : \alpha \in [a,b]$. Then H_1 is rejected if $D^0_{n,\kappa} \geq \sup_{\alpha\in[a,b]} \ D_{n,\kappa}(\xi;\alpha)$.*

3. *We plug in the estimate $\hat{\alpha_n}$ in $D_{n,\kappa}(\xi;\hat{\alpha}_n)$ and H_1 is rejected if $D^0_{n,\kappa} \geq D_{n,\kappa}(\xi;\hat{\alpha}_n)$.*

Though procedure 1 is logically correct for finite sample, it has the drawback that the null hypothesis with small true characteristic exponent α_0 may be rejected by comparing $D^0_{n,\kappa}$ to percentage points near $\alpha = 2$ (See Figures 7-10). Therefore procedure 2 and 3 are also worth considering. From the viewpoint of asymptotic theory we can use procedure 3. We may use Procedure 2 considering the standard error in $\hat{\alpha}_n$.

A Proof of Theorem 2.2

For simplicity we definite some constants and functions.

$$w''(x) := d^2w(x)/dx^2, \quad w_1(t) := w'(|t|)|t| + w(t), \quad w_2(t) := \alpha|t|^\alpha w(t),$$

$$c_1 = \frac{1}{2}\int_{-\infty}^{\infty} e^{-2|t|^\alpha} w_1(t)dt, \quad c_2 = \int_{-\infty}^{\infty} e^{-2|t|^\alpha} w_2(t)dt.$$

Expanding the estimating equation $Q_\theta(\theta) = (Q_\mu(\theta), Q_\sigma(\theta), Q_\alpha(\theta)) = 0$ around the true parameter $\theta_0 = (0,1,\alpha)$, we have

$$Q_\theta(\theta_0) + \frac{\partial Q_\theta(\theta^*)}{\partial\theta}(\hat{\theta}_n - \theta_0) = 0,$$

where θ^*_n is some value between θ_0 and $\hat{\theta}_n$. We can write

$$\begin{aligned}
\sqrt{n}Q_\mu(\theta) &= -\frac{1}{\sqrt{n}}\sum_{j=1}^n g_1(X_j), \\
\sqrt{n}Q_\sigma(\theta) &= \frac{\sqrt{n}}{2}\left\{\frac{1}{n^2}\sum_{j,k=1}^n h_1(X_j,X_k) - \frac{1}{n}\sum_{j=1}^n 2h_2(X_j)\right\}, \\
\sqrt{n}Q_\alpha(\theta) &= \frac{1}{\sqrt{n}}\sum_{j=1}^n g_2(X_j),
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
g_1(x) &= \int_{-\infty}^{\infty} \sin(tx)te^{-|t|^\alpha} w(t)dt, \\
g_2(x) &= \int_{-\infty}^{\infty} \left\{\cos(tx) - e^{-|t|^\alpha}\right\}\left(|t|^\alpha \log|t| e^{-|t|^\alpha}\right)w(t)dt, \\
h_1(x_1,x_2) &= \int_{-\infty}^{\infty} \cos(t(x_1-x_2))w_1(t)dt, \\
h_2(x) &= \int_{-\infty}^{\infty} \cos(tx)e^{-|t|^\alpha}(w_1(t) - w_2(t))dt - c_1 + c_2,
\end{aligned}$$

$2\sqrt{n}Q_\sigma(\theta)$ can be expressed in the form of a U-statistic

$$2\sqrt{n}Q_\sigma(\theta) = \sqrt{n}\left\{U_n + \frac{h_1(X_1,X_1)}{n} - \frac{1}{n^2(n-1)}\sum_{j<k}^{n} 2h_1(X_j,X_k)\right\} = \sqrt{n}U_n + r_n, \qquad r_n \xrightarrow{P} 0,$$

where

$$U_n = \frac{2}{n(n-1)}\sum_{1\le j<k\le n}^{n}\{h_1(X_j,X_k) - h_2(X_j) - h_2(X_k)\} = \binom{n}{2}^{-1}\sum_{1\le j<k\le n}^{n} h(X_j,X_k).$$

By standard argument on U-statistics (Chapter 3 of Maesono (2001), Chapter 5 of Serfling (1980)) we only need to evaluate

$$a(x_1) = \mathrm{E}[h(X_1,X_2) \mid X_1 = x_1],$$

since

$$\sqrt{n}U_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} 2a(X_j) + r_n, \qquad r_n \xrightarrow{P} 0.$$

Calculating

$$\mathrm{E}[h_1(X_1,X_2)|X_1 = x_1] = \int_{-\infty}^{\infty}\cos(tx_1)e^{-|t|^\alpha}w_1(t)dt$$

and $\mathrm{E}[h_2(X_2)|X_1 = x_1] = \mathrm{E}[h_2(X_2)] = c_1$, it can be shown that $a(x_1)$ is written as

$$a(x_1) = \int_{-\infty}^{\infty}\cos(tx_1)e^{-|t|^\alpha}w_2(t)dt - c_2.$$

After showing the convergence of second derivatives $\partial Q_\theta(\theta^*)/\partial\theta \xrightarrow{P} A$ we can obtain AL representations,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = -A^{-1}\sqrt{n}Q_\theta(\theta_0).$$

The proof of $\partial Q(\theta^*)/\partial\theta \xrightarrow{P} A$ is as follows. As before the derivatives are evaluated at $(\mu,\sigma) = (0,1)$. Write

$$\begin{aligned}
\frac{\partial Q_\mu(\theta)}{\partial\mu} &= \frac{1}{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}\cos(tx_j)t^2e^{-|t|^\alpha}w(t)dt,\\
\frac{\partial Q_\sigma(\theta)}{\partial\sigma} &= \frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{1}{n^2}\sum_{j,k}\cos(t(x_j-x_k)) - \frac{2}{n}\sum_{j=1}^{n}\cos(tx_j)e^{-|t|^\alpha} + e^{-2|t|^\alpha}\right](w''(t)|t|^2 + 2w'(|t|)|t|)dt\\
&\quad +\frac{1}{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}(\cos(tx_j) - e^{-|t|^\alpha})e^{-|t|^\alpha}|t|^\alpha\left\{(-\alpha^2|t|^\alpha + \alpha^2 + \alpha)w(t) + 2\alpha w'(t)|t|\right\}dt\\
&\quad +\alpha^2\int_{-\infty}^{\infty}e^{-2|t|^\alpha}|t|^{2\alpha}w(t)dt,\\
\frac{\partial Q_\sigma(\theta)}{\partial\alpha} &= \frac{1}{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}(\cos(tx_j) - e^{-|t|^\alpha})e^{-|t|^\alpha}|t|^\alpha\left[\{\log|t|(-\alpha|t|^\alpha + \alpha + 1) + 1\}w(t) + \log|t|w'(t)|t|\right]dt\\
&\quad +\alpha\int_{-\infty}^{\infty}e^{-2|t|^\alpha}|t|^{2\alpha}\log|t|w(t)dt,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial Q_\alpha(\theta)}{\partial \alpha} &= \frac{1}{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty}(\cos(tx_j)-e^{-|t|^\alpha})(1-|t|^\alpha)e^{-|t|^\alpha}|t|^\alpha(\log|t|)^2 w(t)dt \\
&\quad+\int_{-\infty}^{\infty} e^{-2|t|^\alpha}|t|^{2\alpha}(\log|t|)^2 w(t)dt.
\end{aligned}
$$

Making use of $\mathrm{E}[\cos(tX)-e^{-|t|^\alpha}]=0$ and by Fubini's theorem we can calculate their expectations,

$$
\begin{aligned}
\mathrm{E}\left[\frac{\partial Q_\mu(\theta)}{\partial \mu}\right] &= \int_{-\infty}^{\infty} e^{-2|t|^\alpha} t^2 w(t)dt, \\
\mathrm{E}\left[\frac{\partial Q_\sigma(\theta)}{\partial \sigma}\right] &= \frac{1}{2n}\int_{-\infty}^{\infty}(1-e^{-2|t|^\alpha})(w''(t)|t|^2+2w'(t)|t|)dt \\
&\quad +\alpha^2\int_{-\infty}^{\infty} e^{-2|t|^\alpha}|t|^{2\alpha} w(t)dt, \\
\mathrm{E}\left[\frac{\partial Q_\sigma(\theta)}{\partial \alpha}\right] &= \alpha\int_{-\infty}^{\infty} e^{-2|t|^\alpha}|t|^{2\alpha}\log|t| w(t)dt, \\
\mathrm{E}\left[\frac{\partial Q_\alpha(\theta)}{\partial \alpha}\right] &= \int_{-\infty}^{\infty} e^{-2|t|^\alpha}|t|^{2\alpha}(\log|t|)^2 w(t)dt.
\end{aligned}
$$

$\mathrm{E}[\partial Q_{\theta_i}(\theta)/\partial\theta_j]=0$ for other parameters θ_i since the density is symmetric. By the weak law of large numbers and continuity of integral about parameters we can finish the proof. □

B Proofs of Section 2.2

B.1 Proof of Theorem 2.3

The idea of proofs is essentially the same as those of Gürtler and Henze (2000) based on the original proof of Csörgő (1983). However, since parameter α is additionally estimated and stable distribution $f(x;\alpha_0)$ dose not have the α-moment $\mathrm{E}[|X|^\alpha]$, $\alpha>\alpha_0>0$, we reproduce here the outline of the whole proof. Note that the kernel $k(s,t)$ is changed from Gürtler and Henze (2000). Before considering Fréchet space $C(\mathbf{R})$, we first assume the restricted space $C(S)$ of continuous functions on a compact subset S with the supremum norm $\|f\|_\infty=\sup_{t\in S}|f(t)|$. Letting $k(x,t)=\cos(tx)+i\sin(tx)$, an alternative representation of $\hat{Z}_n(t)$ is given by

$$
\begin{aligned}
\hat{Z}_n(t) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left\{\cos(tx_j)+i\sin(tx_j)-e^{-|\hat{\sigma}_n t|^{\hat{\alpha}_n}}(\cos(t\hat{\mu_n})+i\sin(t\hat{\mu_n}))\right\} \\
&= \int k(x,t)d\left\{\sqrt{n}\big(F_n(x)-F(x;\hat{\theta}_n)\big)\right\}.
\end{aligned}
$$

This is the form of kernel transformed empirical process. We have to check the condition of (i)* (ii)*, (iv), (v) and (vi) of Csörgő (1983). Condition (i)* is satisfied from the definition of the kernel $k(x,t)=$

$\cos(tx) + i\sin(tx)$. Condition (ii)* is easy following Gürtler and Henze (2000). For $0 < \epsilon < \alpha_0$,

$$
\begin{aligned}
|\, k(x,s) - k(x,t)\,| &= \sqrt{(\cos(sx) - \cos(tx))^2 + (\sin(sx) - \sin(tx))^2} \\
&= \sqrt{2}\sqrt{1 - \cos\left((s-t)x\right)} \\
&= 2\left|\,\sin((s-t)x/2)\,\right| \\
&\le 4\,|\, s - t\,|^{\epsilon/2}|\, x\,|^{\epsilon/2},
\end{aligned}
$$

and $\mathrm{E}[|X|^{\epsilon} < \infty]$. For condition (iv) we can check that the covariance matrix $\mathrm{E}[l(X)l(X)']$ for MLE and EISE are finite and positive definite from Theorem 2.1 and Theorem 2.2. Condition (v) for EISE is easy because l_θ are bounded and continuously differentiable from Theorem 2.2. However for MLE we need some argument since $h_\theta = f_\theta/f$ of Theorem 2.1 has no explicit form. For any compact set $K \in \mathbf{R}$ if $x \in K$, $f(x;\alpha)$ and $f_\theta(x;\alpha)$ are continuously differentiable with respect to x. Thus h_θ is exists almost everywhere and finite $x \in K$. To see this we differentiate Fourier inversion formula directly and confirm its integrability. Note that the density $f(x;\alpha)$ has no singularity. However as $x \to \infty$ we have to consider the tail orders of $f(x;\alpha)$ and $f_\theta(x;\alpha)$ going to 0. We utilize asymptotic expansions of $f(x;\alpha)$ and $f_\theta(x;\alpha)$ in Matsui and Takemura (2006):

$$
\begin{aligned}
f(x;\alpha) &= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{k!}(-1)^{k-1}\sin(\frac{\pi\alpha k}{2})x^{-k\alpha-1}, \\
f'(x;\alpha) &= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k+2)}{k!}(-1)^{k}\sin(\frac{\pi\alpha k}{2})x^{-k\alpha-2}, \\
f_\alpha(x;\alpha) &= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma'(\alpha k+1)}{(k-1)!}(-1)^{k-1}\sin\left(\frac{\pi\alpha k}{2}\right)x^{-k\alpha-1} \\
&\quad +\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k+1)}{(k-1)!}(-1)^{k-1}\left[\frac{\pi}{2}\cos\left(\frac{\pi\alpha k}{2}\right) - \log x\ \sin\left(\frac{\pi\alpha k}{2}\right)\right]x^{-k\alpha-1}.
\end{aligned}
$$

From expansions and the following relations

$$
f_\mu(x;\alpha) = -f'(x;\alpha), \quad f_\sigma(x;\alpha) = -f(x;\alpha) - xf(x;\alpha),
$$

we can confirm that l_θ's exist almost everywhere and finite except for h_α at $x = \infty$ since $h_\alpha(x) = O(\log x)$. From the expansions we see that another condition of (v), i.e.

$$
V_l(u) = \sup_{|x|\le u}\left\{\left|l(x;\theta^0)\right| + \left|\frac{\partial}{\partial x}l(x;\theta^0)\right|\right\} < \infty, \qquad \forall u > 0, \tag{B.1}
$$

is also satisfied. Note that in the proof of Theorem at p.527 of Csörgő (1983) non-existence of l_θ at $x = \infty$ is permissible. Therefore condition (v) holds.

Concerning condition (vi), for symmetric stable distributions elements of $\nabla_\theta\Phi(t;\theta) = (\Phi_\mu(t;\theta), \Phi_\sigma(t;\theta), \Phi_\alpha(t;\theta))$ are written as

$$
\begin{aligned}
\Phi_\mu(t;\theta) &= \frac{\partial\Phi(t;\alpha,\mu,\sigma)}{\partial\mu} = \{-\sin(\mu t) + i\cos(\mu t)\}te^{-|\sigma t|^{\alpha}}, \\
\Phi_\sigma(t;\theta) &= \frac{\partial\Phi(t;\alpha,\mu,\sigma)}{\partial\sigma} = -\{\cos(\mu t) + i\sin(\mu t)\}e^{-|\sigma t|^{\alpha}}|\sigma t|^{\alpha}\alpha/\sigma, \\
\Phi_\alpha(t;\theta) &= \frac{\partial\Phi(t;\alpha,\mu,\sigma)}{\partial\alpha} = -\{\cos(\mu t) + i\sin(\mu t)\}e^{-|\sigma t|^{\alpha}}|\sigma t|^{\alpha}\log|\sigma t|.
\end{aligned}
$$

Putting $(\mu, \sigma, \alpha) = \theta_0 = (0, 1, \alpha)$ we obtain

$$\nabla_\theta \Phi(t; \theta_0) = (\Phi_\mu(t; \alpha), \Phi_\sigma(t; \alpha), \Phi_\alpha(t; \alpha)) = \left(ite^{-|t|^\alpha}, -e^{-|t|^\alpha}|t|^\alpha \alpha, -e^{-|t|^\alpha}|t|^\alpha \log |t|\right).$$

Because $\nabla_\theta \Phi(t; \theta)$ is continuous and bounded if $(t, \theta) \in S \times \Theta_0$, where Θ_0 is a closed parameter space sufficiently near θ_0, the condition is satisfied. Therefore the weak convergence of $\hat{Z}_n(t)$ to a zero mean Gaussian process Z is proved in the space $(C(S), \|\cdot\|_\infty)$. Since the compact set S is arbitrary, the space $(C(S), \|\cdot\|_\infty)$ can be extended to Fréchet space $C(\mathbf{R})$ easily.

The kernel transform of AL representations are given as follows. Write $F_0(x) = F(x, \theta_0)$ for simplicity.
1. MLE :

$$\int_{-\infty}^{\infty} k(x, s) l_\theta(x) dF_0(x) = I^{-1}(\theta) \nabla_\theta \Phi(s; \theta). \tag{B.2}$$

2. EISE :

$$\int k(x, s) l_\theta(x) dF_0(x) = A^{-1} \int k(x, s) h_\theta(x) dF_0(x), \tag{B.3}$$

where

$$\begin{aligned}
\int k(x, s) h_\mu(x) dF_0(x) &= i \int_{-\infty}^{\infty} e^{-|s-u|^\alpha - |u|^\alpha} u w(u) du, \\
\int k(x, s) h_\sigma(x) dF_0(x) &= B_\sigma e^{-|s|^\alpha} - \alpha \int_{-\infty}^{\infty} e^{-|s-u|^\alpha - |u|^\alpha} |u|^\alpha w(u) du, \\
\int k(x, s) h_\alpha(x) dF_0(x) &= B_\alpha e^{-|s|^\alpha} - \int_{-\infty}^{\infty} e^{-|s-u|^\alpha - |u|^\alpha} |u|^\alpha \log |u| w(u) du.
\end{aligned}$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of $\mathbf{R}^3$. Write

$$\begin{aligned}
\Phi(t; \hat{\theta}_n) &= \Phi(t; \theta_0) - \langle \hat{\theta}_n - \theta_0, \nabla_\theta \Phi(t, \theta_n^*) \rangle \\
&= \int k(x, t) dF_0(x) - \langle \hat{\theta}_n - \theta_0, \nabla_\theta \Phi(t, \theta_n^*) \rangle,
\end{aligned}$$

where θ_n^* is some value between θ_0 and θ_n. Note that $\theta_n^* \xrightarrow{P} \theta_0$. Now replace $\sqrt{n}(\hat{\theta}_n - \theta_0)$ by its AL representations. Then $\hat{Z}_n(t)$ is written as

$$\begin{aligned}
\hat{Z}_n(t) &= \int k(x, t) d\left\{\sqrt{n}\big(F_n(x) - F_0(x)\big)\right\} - \left\langle \sqrt{n}(\hat{\theta}_n - \theta_0), \nabla_\theta \Phi(t; \theta_n^*) \right\rangle \\
&= Z_n^*(t) + \Delta_n^{(2)}(t) + \Delta_n^{(3)}(t),
\end{aligned}$$

where

$$Z_n^*(t) := \int k(x, t) d\left\{\sqrt{n}\big(F_n(x) - F_0(x)\big)\right\} - \left\langle \frac{1}{\sqrt{n}} \sum_{j=1}^{n} l_\theta(x_j), \nabla_\theta \Phi(t; \theta_0) \right\rangle.$$

Z_n^* also converges to Z. The remainder terms $\Delta_n^{(2)}$ and $\Delta_n^{(3)}$ are defined by

$$\begin{aligned}
\Delta_n^{(2)} &:= \left\langle \sqrt{n}(\hat{\theta}_n - \theta_0), \nabla_\theta \Phi(t; \theta_0) - \nabla_\theta \Phi(t; \theta_n^*) \right\rangle, \\
\Delta_n^{(3)} &:= -\langle \epsilon_n, \nabla_\theta \Phi(t; \theta_0) \rangle, \qquad \epsilon_n = (r_{n1}, r_{n2}, r_{n3})'.
\end{aligned}$$

These remainder terms satisfy $\sup_{t\in S}|\Delta_n^{(2)}| \xrightarrow{P} 0$, and $\sup_{t\in S}|\Delta_n^{(3)}| \xrightarrow{P} 0$ by conditions (iv) and (vi) of Csörgő (1983). The asymptotic process Z has an alternative expression

$$Z(t) = \int k(x,t)dB_{F_0}(x) - \left\langle \int l_\theta(x)dB_{F_0}(x), \nabla_\theta \Phi(t;\theta_0) \right\rangle,$$

where $B_{F_0}(x)$ is the Brownian bridge corresponding to the distribution function F_0, having covariance function $\mathrm{E}[B_{F_0}(s)B_{F_0}(t)] = F_0(s\wedge t) - F_0(s)F_0(t)$. Z^* and Z have the same covariance function

$$\begin{aligned} \text{(B.4)}\quad \Gamma(s,t) &= \Phi(s-t;\theta_0) - \Phi(s;\theta_0)\overline{\Phi(t;\theta_0)} + \Phi(s,\theta_0)'\mathrm{E}[l_\theta(X_1)l_\theta(X_1)']\overline{\Phi(t,\theta_0)} \\ &\quad - \left\langle \nabla_\theta\Phi(s;\theta_0), \int \overline{k(x,t)}l_\theta(x)dF_0(x)\right\rangle - \left\langle \overline{\nabla_\theta\Phi(t;\theta_0)}, \int k(x,s)l_\theta(x)dF_0(x)\right\rangle. \end{aligned}$$

Note

$$\int k(x,s)\overline{k(x,t)}dF_0(x) = \int k(x,s-t)dF_0(x) = \Phi(s-t;\theta_0).$$

Evaluating (B.4) for the case of MLE and EISE using (B.2) and (B.3) proves Theorem 2.3. □

B.2 Proof of theorem 2.4

We have only to show

$$\text{(B.5)}\qquad \nabla_\theta\Phi(s;\theta_0)'I^{-1}(\theta)\overline{\nabla_\theta\Phi(t;\theta_0)} = \left\langle \overline{\nabla_\theta\Phi(t;\theta_0)}, \int k(x,s)l_E(x)dF_0(x)\right\rangle.$$

Because AL representations can be written by $I^{-1}(\theta)\partial\log f(x;\theta_0)/\partial\theta$, their kernel transformations are

$$\begin{aligned} \int k(x,s)l_E(x)dF_0(x) &= I^{-1}(\theta)\int k(x,s)\frac{\partial\log f(x;\theta_0)}{\partial\theta}dF_0(x). \\ &= I^{-1}(\theta)\int k(x,s)\frac{1}{f(x;\theta_0)}\frac{\partial f(x;\theta_0)}{\partial\theta}\ f(x;\theta_0)dx \\ &= I^{-1}(\theta)\frac{1}{2\pi}\int e^{isx}\frac{\partial f(x;\theta_0)}{\partial\theta}dx \\ &= I^{-1}(\theta)\nabla_\theta\Phi(s;\theta_0). \end{aligned}$$

Since both sides of the formula (B.5) are scalars the proof is over. □

B.3 Proof of corollary 2.2.1

Let $\alpha = 1$. Then $I^{-1}(\theta)$ is explicitly written as

$$I^{-1}(\theta) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\frac{12}{\pi^2}(\gamma+\log 2-1)^2 & \frac{12}{\pi^2}(\gamma+\log 2-1) \\ 0 & \frac{12}{\pi^2}(\gamma+\log 2-1) & \frac{12}{\pi^2} \end{pmatrix}.$$

Letting $\alpha = 1$ in (2.18) and replacing $I^{-1}(\theta)$ by the above matrix we can prove the corollary. □

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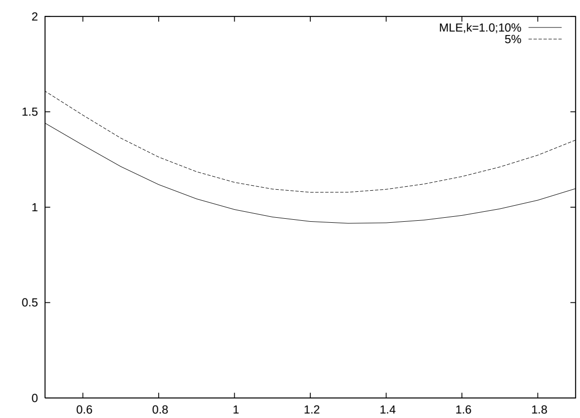

Figure 7: Upper quantiles ($\kappa = 1.0$, H_1)

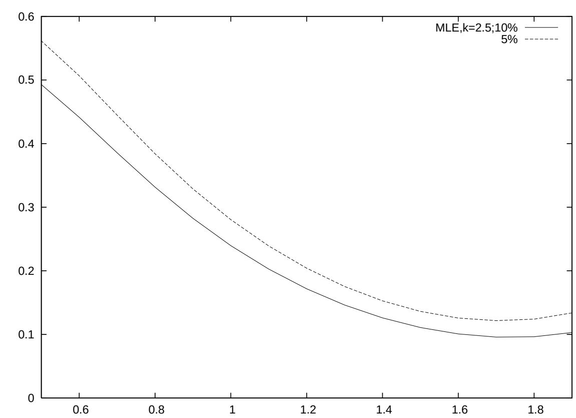

Figure 8: Upper quantiles ($\kappa = 2.5$, H_1)

Table 1: Upper ξ percentage points of D_κ under H_1

α	$\xi\backslash\kappa$	1.0	2.5	5.0	10.0
1.9	0.1	1.097	0.1030	0.00759	0.000973
	0.05	1.352	0.1338	0.01000	0.001267
1.8	0.1	1.037	0.0963	0.00977	0.002283
	0.05	1.273	0.1240	0.01257	0.002984
1.7	0.1	0.991	0.0958	0.01375	0.003974
	0.05	1.211	0.1217	0.01754	0.005183
1.6	0.1	0.957	0.1007	0.01921	0.006077
	0.05	1.161	0.1258	0.02431	0.007895
1.5	0.1	0.933	0.1108	0.02615	0.008650
	0.05	1.122	0.1361	0.03280	0.011180
1.4	0.1	0.918	0.1260	0.03469	0.011770
	0.05	1.094	0.1527	0.04313	0.015120
1.3	0.1	0.915	0.1462	0.04503	0.015530
	0.05	1.078	0.1754	0.05553	0.019813
1.2	0.1	0.925	0.1717	0.05740	0.020042
	0.05	1.078	0.2041	0.07025	0.025370
1.1	0.1	0.948	0.2027	0.07206	0.025434
	0.05	1.095	0.2390	0.08755	0.031916
1.0	0.1	0.988	0.2395	0.08923	0.031846
	0.05	1.130	0.2804	0.10765	0.039574
0.9	0.1	1.044	0.2824	0.10903	0.039422
	0.05	1.186	0.3288	0.13063	0.048454
0.8	0.1	1.118	0.3315	0.13145	0.048287
	0.05	1.262	0.3840	0.15627	0.058625
0.7	0.1	1.213	0.3855	0.15611	0.058531
	0.05	1.362	0.4446	0.18397	0.070081
0.6	0.1	1.325	0.4413	0.18217	0.070175
	0.05	1.482	0.5065	0.21239	0.082722
0.5	0.1	1.441	0.4928	0.20834	0.083189
	0.05	1.609	0.5615	0.23966	0.096393

Table 2: Upper ξ percentage points of D_κ under H_2

α	$\xi \backslash \kappa$	1.0	2.5	5.0	10.0
2.0	0.1	1.216	0.1258	0.00881	0.000241
	0.05	1.499	0.1622	0.01177	0.000335
1.9	0.1	1.150	0.1129	0.00921	0.00142
	0.05	1.413	0.1444	0.01164	0.00179
1.8	0.1	1.110	0.1111	0.01354	0.00329
	0.05	1.357	0.1398	0.01679	0.00416
1.7	0.1	1.080	0.1157	0.01984	0.00566
	0.05	1.313	0.1431	0.02457	0.00713
1.6	0.1	1.058	0.1256	0.02773	0.00858
	0.05	1.277	0.1532	0.03426	0.01076
1.5	0.1	1.044	0.1404	0.03721	0.01211
	0.05	1.249	0.1697	0.04578	0.01514
1.4	0.1	1.037	0.1599	0.04840	0.01636
	0.05	1.231	0.1919	0.05927	0.02037
1.3	0.1	1.039	0.1840	0.06148	0.02144
	0.05	1.222	0.2195	0.07493	0.02660
1.2	0.1	1.051	0.2127	0.07670	0.02748
	0.05	1.225	0.2524	0.09300	0.03396
1.1	0.1	1.074	0.2465	0.09428	0.03464
	0.05	1.242	0.2907	0.11376	0.04262
1.0	0.1	1.111	0.2862	0.11445	0.04307
	0.05	1.276	0.3356	0.13742	0.05273
0.9	0.1	1.159	0.3305	0.13723	0.05291
	0.05	1.322	0.3855	0.16395	0.06443
0.8	0.1	1.226	0.3811	0.16263	0.06426
	0.05	1.389	0.4424	0.19326	0.07779
0.7	0.1	1.310	0.4365	0.19018	0.07714
	0.05	1.476	0.5045	0.22466	0.09278
0.6	0.1	1.412	0.4936	0.21890	0.09144
	0.05	1.583	0.5678	0.25681	0.10919
0.5	0.1	1.517	0.5464	0.24726	0.10689
	0.05	1.696	0.6250	0.28773	0.12661

Table 3: Simulation results of symmetric stable distributions (1000 iterations)

n	$\hat{\alpha}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{I}_{11}$	$\hat{I}_{22}$	$\hat{I}_{33}$	$\hat{I}_{12}$	$\hat{I}_{13}$	$\hat{I}_{23}$
	2.0	0	1	0.5	2.0	∞	0	0	*
50	1.976	0.00014	0.977	0.607	1.875	5.647	-0.029	-0.010	-0.828
100	1.990	0.00017	0.975	0.868	1.402	9.445	0.032	-0.057	-0.594
200	1.994	0.00094	0.977	0.784	1.239	12.77	-0.009	0.078	-0.539
	1.8	0	1	0.4552	1.3898	0.5937	0	0	-0.3138
50	1.818	0.0012	0.991	0.487	1.231	0.676	-0.033	-0.026	-0.267
100	1.822	0.0033	1.002	0.482	1.356	0.584	0.005	-0.016	-0.340
200	1.810	-0.0000	1.000	0.450	1.399	0.603	-0.007	-0.003	-0.323
	1.5	0	1	0.4281	0.9556	0.4737	0	0	-0.2174
50	1.548	-0.0022	1.012	0.3161	0.5796	0.4252	0.005	-0.009	-0.1927
100	1.524	-0.0000	1.000	0.3914	0.9138	0.4278	0.001	-0.010	-0.2291
200	1.510	-0.0003	1.000	0.4028	0.9474	0.4683	-0.018	-0.006	-0.2229
	1.0	0	1	0.5	0.5	0.8590	0	0	-0.1352
50	1.026	-0.0029	0.996	0.4243	0.4877	0.670	-0.004	0.0248	-0.1779
100	1.001	-0.0041	0.988	0.4438	0.4205	0.746	0.007	0.003	-0.1527
200	1.006	-0.0039	1.001	0.4929	0.5013	0.845	-0.003	-0.0251	-0.1534
	0.8	0	1	0.6800	0.3586	1.3928	0	0	-0.0913
50	0.815	-0.0016	1.005	0.5434	0.3243	1.111	-0.01432	0.0015	-0.083
100	0.811	-0.0001	1.003	0.6015	0.3459	1.171	0.00435	-0.0313	-0.097
200	0.805	-0.0009	1.000	0.6232	0.3708	1.303	0.01785	-0.0276	-0.086

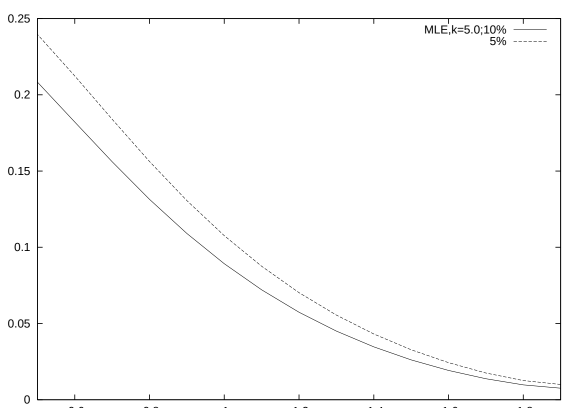

Figure 9: Upper quantiles ($\kappa = 5.0,\ H_1$)

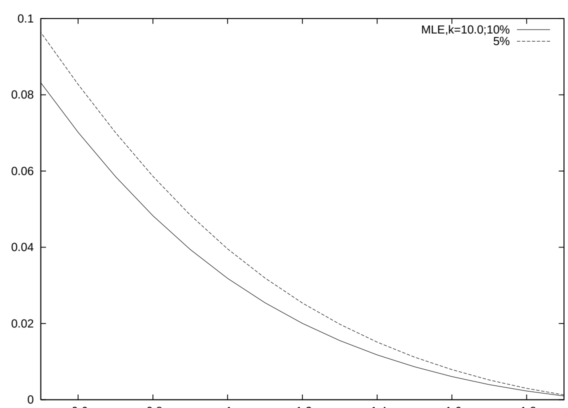

Figure 10: Upper quantiles ($\kappa = 10.0,\ H_1$)

Table 4: Upper 10 percentage points of $D_{100,\kappa}$ under H_1

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.037	0.1037	0.01271	0.00326
	1.037	0.0963	0.00977	0.00228
1.5	0.953	0.1213	0.03061	0.00996
	0.933	0.1108	0.02615	0.00865
1.0	1.032	0.2541	0.09450	0.03336
	0.988	0.2395	0.08923	0.03185

Table 5: Upper 5 percentage points of $D_{100,\kappa}$ under H_1

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.279	0.1334	0.01681	0.00451
	1.273	0.1240	0.01257	0.00298
1.5	1.141	0.1504	0.03968	0.01348
	1.122	0.1361	0.03280	0.01118
1.0	1.216	0.3059	0.11400	0.04230
	1.130	0.2804	0.10765	0.03957

Table 6: Upper 10 percentage points of $D_{200,\kappa}$ under H_1

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.029	0.1003	0.01119	0.00279
	1.037	0.0963	0.00977	0.00228
1.5	0.929	0.1145	0.02807	0.00926
	0.933	0.1108	0.02615	0.00865
1.0	1.006	0.2462	0.09072	0.03233
	0.988	0.2395	0.08923	0.03185

Table 7: Upper 5 percentage points of $D_{200,\kappa}$ under H_1

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.290	0.1292	0.01467	0.00386
	1.273	0.1240	0.01257	0.00298
1.5	1.125	0.1419	0.03575	0.01222
	1.122	0.1361	0.03280	0.01118
1.0	1.161	0.2887	0.10938	0.04015
	1.130	0.2804	0.10765	0.03957

Table 8: Upper 10 percentage points of $D_{100,\kappa}$ under H_2

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.100	0.1126	0.01415	0.00356
	1.110	0.1111	0.01354	0.00329
1.5	1.030	0.1403	0.03709	0.01202
	1.044	0.1404	0.03721	0.01211

Table 9: Upper 5 percentage points of $D_{100,\kappa}$ under H_2

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.333	0.1397	0.01878	0.00533
	1.357	0.1398	0.01679	0.00416
1.5	1.220	0.1690	0.04710	0.01590
	1.249	0.1697	0.04578	0.01514

Table 10: Upper 10 percentage points of $D_{200,\kappa}$ under H_2

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.077	0.1108	0.01369	0.00332
	1.110	0.1111	0.01354	0.00329
1.5	1.037	0.1414	0.03754	0.01204
	1.044	0.1404	0.03721	0.01211

Table 11: Upper 5 percentage points of $D_{200,\kappa}$ under H_2

$\alpha \setminus \kappa$	1.0	2.5	5.0	10.0
1.8	1.368	0.1409	0.01818	0.00470
	1.357	0.1398	0.01679	0.00416
1.5	1.244	0.1695	0.04620	0.01561
	1.249	0.1697	0.04578	0.01514

Table 12: Power of $D_{n,\kappa}$ under H_2 ($\alpha = 1.5$, significance levels $\xi = 0.1,\ 0.05,\ n = 100$)

ξ	0.1				0.05			
κ	1.0	2.5	5.0	10.0	1.0	2.5	5.0	10.0
$N(0,2)$	44	68	76	45	30	49	45	2
$t(1)$	82	93	96	95	75	90	92	92
$t(2)$	16	20	17	15	9	12	10	8
$t(3)$	11	10	5	3	6	6	2	1
$t(4)$	12	12	9	3	7	6	3	0
$t(5)$	14	15	15	5	8	9	5	0
$t(10)$	26	40	42	15	14	22	16	0

Table 13: Power of $D_{n,\kappa}$ under H_2 ($\alpha = 1.5$, significance levels $\xi = 0.1,\ 0.05,\ n = 200$)

ξ	0.1				0.05			
κ	1.0	2.5	5.0	10.0	1.0	2.5	5.0	10.0
$N(0,2)$	79	99	100	100	63	96	99	98
$t(1)$	98	99	100	100	96	99	100	100
$t(2)$	24	26	21	16	16	18	14	10
$t(3)$	12	10	10	9	6	6	5	3
$t(4)$	16	24	32	27	8	12	16	8
$t(5)$	23	40	50	47	12	26	33	19
$t(10)$	48	83	93	92	34	66	84	68

Table 14: Power of $D_{n,\kappa}$ under H_2 ($\alpha = 1.8$, significance levels $\xi = 0.1,\ 0.05,\ n = 100$)

ξ	0.1				0.05			
κ	1.0	2.5	5.0	10.0	1.0	2.5	5.0	10.0
$N(0,2)$	14	14	3	0	8	8	1	0
$t(1)$	98	100	100	100	96	100	100	100
$t(2)$	55	72	77	70	41	60	65	55
$t(3)$	28	35	34	22	20	25	23	12
$t(4)$	16	18	14	6	10	10	6	3
$t(5)$	11	12	9	4	6	6	3	1
$t(10)$	9	10	3	0	4	5	1	0

Table 15: Power of $D_{n,\kappa}$ under H_2 ($\alpha = 1.8$, significance levels $\xi = 0.1,\ 0.05,\ n = 200$)

ξ	0.1				0.05			
κ	1.0	2.5	5.0	10.0	1.0	2.5	5.0	10.0
$N(0,2)$	20	23	18	0	10	12	4	0
$t(1)$	99	100	100	100	97	99	100	100
$t(2)$	83	94	94	89	72	88	89	80
$t(3)$	45	54	49	33	28	37	37	21
$t(4)$	24	25	18	7	13	15	9	4
$t(5)$	16	17	9	3	7	9	4	1
$t(10)$	13	12	9	1	5	6	2	0